Large-N saddle points

1981 J. Phys. A: Math. Gen. 14 1925

(http://iopscience.iop.org/0305-4470/14/8/017)

J. Phys. A: Math. Gen. 14 (1981) 1925–1929. Printed in Great Britain

Large-N saddle points

C Aragão de Carvalho and V A Fateev†
CERN, Geneva, Switzerland

Received 16 January 1981

Abstract. We use the saddle-point method to study the large-N limit in quantum mechanics. In a model with O(N) symmetry this yields the correct answer for the ground state energy. However, in a (quaternionic) matrix model with Sp(N) symmetry that is not the case.

1. Introduction

In this work we examine the large-N limit in quantum mechanics by the saddle-point method. We present the calculation of the ground state energy of a $N \times N$ quaternionic matrix model for which the saddle-point result (to leading order in N) does not yield the correct answer. Corrections to the saddle-point value prove to be of the same order in N. We contrast this with the case of an O(N)-symmetric model where the saddle-point result does provide the correct answer and corrections are suppressed by one power of N. In both cases we use the exactly soluble harmonic oscillator potential to establish our conclusions.

In § 2 the O(N) model is discussed. In § 3 we apply the same approach to the matrix model and point out the differences between the two cases. Our results are relevant for recent discussions (Jevicki 1980, Levine 1980) of the role of classical solutions in the large-N limit of various field theories in which zero-dimensional matrix models are used as prototypes.

2. The O(N) model

Consider the O(N)-invariant classical Euclidean Lagrangian given by

$$L = \tfrac{1}{2} \sum_{i=1}^{N} \dot{q}_i^2 + V(\boldsymbol{q}^2). \tag{1}$$

The quantum Hamiltonian is obviously

$$H = -\tfrac{1}{2}\nabla^2 + V(\boldsymbol{q}^2). \tag{2}$$

The Laplace operator can be decomposed into radial and angular parts

$$\nabla^2 \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial q_i^2} = \frac{\partial^2}{\partial r^2} + \frac{N-1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\nabla_\Omega^2 = H_0 + \frac{1}{r^2}\nabla_\Omega^2 \tag{3}$$

† On leave from the Landau Institute of Theoretical Physics, Moscow, USSR.

where $r = (q^2)^{1/2}$ and ∇^2_Ω is the Beltrami operator. Since the ground state of the model is O(N) symmetric, the Hamiltonian for that state is just H_0. The ground state energy is given by

$$E_0 = \min \frac{\int [\mathrm{d}^N q] \langle \psi_0 | H_0 | \psi_0 \rangle}{\int [\mathrm{d}^N q] \langle \psi_0 | \psi_0 \rangle} \tag{4}$$

where $[\mathrm{d}^N q] \equiv r^{N-1} \mathrm{d}r\, \mathrm{d}\Omega$. The angular integrations cancel out, and if we define $\chi_0 \equiv r^{(N-1)/2} \psi_0$, the equation for χ_0 is

$$-\frac{1}{2} \frac{\partial^2 \chi_0}{\partial r^2} + \frac{N^2 - 1}{8r^2} \chi_0 + V(r^2) \chi_0 = E_0 \chi_0. \tag{5}$$

If N is large we may neglect the derivative term, and the minimisation condition just amounts to finding the extrema of the effective potential:

$$V_{\mathrm{eff}}(r^2) = (N^2 - 1)/8r^2 + V(r^2). \tag{6}$$

For a harmonic potential, $V(r^2) = \frac{1}{2} r^2$, we obtain

$$\partial V_{\mathrm{eff}} / \partial r |_{r_0} = 0 \Rightarrow r_0 = \tfrac{1}{2}(N^2 - 1)^{1/2}, \qquad V_{\mathrm{eff}}(r_0) = V_{\min} = \tfrac{1}{2}(N^2 - 1)^{1/2}. \tag{7}$$

For large N this agrees with the exact answer $E_0 = N/2$. The corrections to the saddle-point result can be obtained by expanding V_{eff} around r_0; since $(\partial^2 V_{\mathrm{eff}} / \partial r^2)|_{r_0}$ is O(1), they will be reduced by one power of N with respect to the saddle-point value.

3. The matrix model

Let Q be a $N \times N$ matrix whose entries are quaternions:

$$Q_{ij} = Q^\alpha_{ij} e_\alpha = Q^0_{ij} \mathbb{1} + Q^a_{ij} e_a, \qquad Q^\alpha_{ij} \in \mathbb{C},$$

$$e_1 = \begin{bmatrix} \mathrm{i} & 0 \\ 0 & \mathrm{i} \end{bmatrix}, \qquad e_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad e_3 = \begin{bmatrix} 0 & -\mathrm{i} \\ -\mathrm{i} & 0 \end{bmatrix}, \qquad e_a e_b = \varepsilon_{abc} e_c.$$

Any $2N \times 2N$ complex matrix can be written in this form. Let us define the quaternionic operations of

$$\begin{aligned} &\text{conjugation:} && \bar{Q}_{ij} \equiv Q^0_{ij} \mathbb{1} - Q^a_{ij} e_a \\ &\text{complex conjugation:} && Q^*_{ij} \equiv Q^{0*}_{ij} \mathbb{1} + Q^{a*}_{ij} e_a, \\ &\text{Hermitian conjugation:} && Q^+_{ij} \equiv Q^{0*}_{ij} \mathbb{1} - Q^{a*}_{ij} e_a = \bar{Q}^*_{ij}. \end{aligned}$$

Using these, one can define the quaternion matrix operations of

$$\begin{aligned} &\text{Hermitian conjugation } (Q \to Q^+): && (Q^+)_{ij} \equiv Q^+_{ji}, \\ &\text{duality } (Q \to \tilde{Q}): && (\tilde{Q})_{ij} \equiv \bar{Q}_{ji}. \end{aligned}$$

Matrices which are Hermitian ($Q = Q^+$) and self-dual ($Q = \tilde{Q}$) are of course quaternion real ($Q^\alpha_{ij} \in \mathbb{R}$). Furthermore, it can be shown (Mehta 1967) that for such matrices, there exists a symplectic matrix S such that

$$Q = S \Lambda S^{-1}, \qquad S \in \mathrm{Sp}(N),$$

where Λ is diagonal, real and scalar:

$$\Lambda_{ij} = \Lambda_i \delta_{ij} = (\lambda_i e_0)\delta_{ij}, \qquad \lambda_i \in \mathbb{R}, \qquad e_0 \equiv \mathbb{1}.$$

Consider now the $\mathrm{Sp}(N)$ invariant Euclidean Lagrangian given by

$$L = \tfrac{1}{2}\mathrm{Tr}\,\dot{Q}^2 + \mathrm{Tr}\,V(Q^2). \tag{8}$$

The trace is taken with respect to the matrix indices as well as the 2×2 quaternions. The quantum Hamiltonian is then

$$H = -\tfrac{1}{4}\nabla^2 + \mathrm{Tr}\,V(Q^2), \tag{9a}$$

$$\nabla^2 \equiv \sum_{i=1}^{N} \frac{\partial^2}{\partial Q_{ii}^{02}} + \frac{1}{2}\sum_{i<j}^{N}\left(\frac{\partial^2}{\partial Q_{ij}^{02}} + \frac{\partial^2}{\partial Q_{ij}^{a2}}\right). \tag{9b}$$

Since Q can be diagonalised by a $\mathrm{Sp}(N)$ transformation, the Laplace operator can be expressed (Mehta 1967) as the sum of a term involving the eigenvalues λ_i plus an 'angular' part:

$$\nabla^2 \equiv \frac{1}{\Delta^4}\sum_{i=1}^{N}\frac{\partial}{\partial\lambda_i}\left(\Delta^4\frac{\partial}{\partial\lambda_i}\right) + \frac{1}{\Delta^4}\nabla_\Omega^2, \qquad \Delta \equiv \prod_{i<j}^{N}(\lambda_i - \lambda_j). \tag{10}$$

The ground state is $\mathrm{Sp}(N)$ symmetric, so that we can restrict ourselves to the 'radial' term H_0. Its wavefunction ψ_0 is symmetric in the λ_i. Its energy is given by (4) with $[\mathrm{d}^N q]$ replaced by $[\mathrm{d}Q] \equiv \Pi_i\,\mathrm{d}\lambda_i\,\Delta^4\,\mathrm{d}\Omega$. Angular integrations will once again cancel out and the Δ^4 can be absorbed into $\chi_0 \equiv \Delta^2\psi_0$. We note that χ_0 is also symmetric in the λ_i (this would not be so if we had $\mathrm{U}(N)$ rather than $\mathrm{Sp}(N)$ invariance), so that we can try to apply the ordinary saddle-point method. The equation for χ_0 is simply

$$-\frac{1}{4}\sum_{i=1}^{N}\frac{\partial^2\chi_0}{\partial\lambda_i^2} + \left(\sum_{i<j}^{N}\frac{1}{(\lambda_i-\lambda_j)^2} + V(\lambda_i)\right)\chi_0 = E_0\chi_0. \tag{11}$$

In order to arrive at (11) we have used

$$\frac{\partial\ln\Delta}{\partial\lambda_k} = \frac{1}{\Delta}\frac{\partial\Delta}{\partial\lambda_k} = \sum_{\substack{i=1\\ i\neq k}}^{N}\frac{1}{(\lambda_k-\lambda_i)}, \tag{12a}$$

$$\sum_{i<k}^{N}\frac{1}{(\lambda_k-\lambda_i)^2} = \frac{1}{2}\sum_{k=1}^{N}\left(\sum_{\substack{i=1\\ i\neq k}}^{N}\frac{1}{(\lambda_k-\lambda_i)}\right)^2. \tag{12b}$$

If we were to proceed as in the case of the $\mathrm{O}(N)$ model, the derivative term would be neglected and the problem would reduce to extremising the effective potential:

$$V_{\mathrm{eff}}(\lambda_i) = \sum_{i<j}^{N}\frac{1}{(\lambda_i-\lambda_j)^2} + V(\lambda_i). \tag{13}$$

Introducing the density of eigenvalues $\varphi(x)$,

$$\varphi(x) \equiv \sum_{i=1}^{N}\delta(x-\lambda_i), \qquad \int_{-\infty}^{\infty}\mathrm{d}x\,\varphi(x) = N, \tag{14}$$

the effective potential can be rewritten as

$$V_{\mathrm{eff}}(\varphi) = \int_{-\infty}^{\infty}\mathrm{d}x\left(\frac{\pi^2}{6}\varphi^3(x) + V(x)\varphi(x)\right) \tag{15}$$

where we have made use of the identity (Mondello and Onofri 1980)

$$\int_{-\infty}^{\infty} \mathrm{d}x\, \varphi(x) \left(\mathrm{P} \int_{-\infty}^{\infty} \mathrm{d}y \frac{\varphi(y)}{x-y} \right)^2 = \frac{\pi^2}{3} \int_{-\infty}^{\infty} \mathrm{d}x\, \varphi^3(x). \tag{16}$$

From the stationarity requirement $(\delta V_{\mathrm{eff}}/\delta\varphi)|_{\varphi_0} = 0$, subject to the constraint (14), one obtains

$$\varphi_0(x) = (\sqrt{2}/\pi)(\mu - V(x))^{1/2}\theta[\mu - V(x)] \tag{17}$$

where μ is a Lagrange multiplier. Inserting this saddle-point expression back into the constraint equation yields the value of μ as a function of N and allows us to compute $V_{\min} = V_{\mathrm{eff}}(\varphi_0)$.

We can easily show that in the present case the saddle-point result is incorrect. We restrict our attention to the harmonic oscillator potential $V(\lambda_i) = \Sigma_{i=1}^{N} \lambda_i^2$. If we carry out the procedure outlined in the preceding paragraph, we obtain $\mu = \sqrt{2}N$ and

$$V_{\min} = V_{\mathrm{eff}}(\varphi_0) = \int_{-\infty}^{\infty} \mathrm{d}x \left(\frac{\pi^2}{6} \varphi_0^3(x) + x^2 \varphi_0(x) \right) = \frac{N^2}{\sqrt{2}}. \tag{18}$$

However, from the exact ground state solution,

$$\psi_0 = C \exp\left(-\tfrac{1}{2} \operatorname{Tr} Q^2\right) = C \exp\left(- \sum_{i=1}^{N} \lambda_i^2 \right), \tag{19}$$

we can compute E_0, either by direct use of equation (11) or simply by noting that we just have a collection of $N(2N-1)$ uncoupled oscillators (the number of independent Q_{ij}). The answer is

$$E_0 = \tfrac{1}{2}N(2N-1) = N^2 - \tfrac{1}{2}N, \tag{20}$$

which is clearly in disagreement with the saddle-point result. It is not justified to leave out the derivative term, since it contributes to the same order in N as the terms we have kept. It is instructive to go back to (13) and work directly in terms of the eigenvalues $\{\lambda_i\}$. From the inequality

$$\sum_{k=1}^{N} \Omega_k^2 \equiv \sum_{k=1}^{N} \left(\frac{1}{\sqrt{2}} \sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{1}{\lambda_k - \lambda_i} - \lambda_k \right)^2 \geq 0 \tag{21}$$

and using (12*b*) one immediately obtains

$$V_{\mathrm{eff}}(\lambda_i) \geq N(N-1)/\sqrt{2}. \tag{22}$$

The value on the right-hand side is $V_{\min}$, since differentiating (21) with respect to λ_k and looking for the extrema yields

$$\lambda_k^{(0)} = \frac{1}{\sqrt{2}} \sum_{\substack{i=1 \\ i\neq k}}^{N} \frac{1}{\lambda_k^{(0)} - \lambda_i^{(0)}}. \tag{23}$$

This saturates the inequality so that $V_{\min} = V_{\mathrm{eff}}(\lambda_i^{(0)}) = N(N-1)/\sqrt{2}$. Taking a second derivative,

$$M_{lk} \equiv \left.\frac{\partial^2 V_{\mathrm{eff}}}{\partial\lambda_l \partial\lambda_k}\right|_{\lambda^{(0)}} = \sum_{j=1}^{N} \left(\sqrt{2} \frac{\partial\Omega_j}{\partial\lambda_l} \right)_{\lambda^{(0)}} \left(\sqrt{2} \frac{\partial\Omega_j}{\partial\lambda_k} \right)_{\lambda^{(0)}}. \tag{24}$$

Define

$$m_{jl} = m_{lj} = \sqrt{2}\,\frac{\partial \Omega_j}{\partial \lambda_l}\bigg|_{\lambda^{(0)}}$$

$$= -\left[\delta_{jl}\left(\sum_{\substack{i=1\\ i\neq j}}^{N} \frac{1}{(\lambda_j^{(0)} - \lambda_i^{(0)})^2} + \sqrt{2}\right) - \frac{1-\delta_{jl}}{(\lambda_j^{(0)} - \lambda_l^{(0)})^2}\right]. \tag{25}$$

Then (24) is just $M = m^2$. If we rescale the $\lambda_k^{(0)}$, so that $\bar{\lambda}_k = 2^{1/4}\lambda_k^{(0)}$, equation (23) can be satisfied if we choose for the $\bar{\lambda}$ the zeros of the Nth-order Hermite polynomial (Calogero 1978). Furthermore, as shown by Calogero (1978), the matrix

$$\delta_{jl} \sum_{\substack{i=1\\ i\neq j}}^{N} \frac{1}{(\bar{\lambda}_j - \bar{\lambda}_i)^2} - \frac{1-\delta_{jl}}{(\bar{\lambda}_j - \bar{\lambda}_l)^2} \tag{26}$$

has integer eigenvalues $n = 0, 1, \ldots, N-1$. Thus, the eigenvalues of m are given by $n' = -\sqrt{2}(n+1)$. The matrix $M = m^2$ has therefore eigenvalues $2(n+1)^2$. Since these are the squares of the normal frequencies of oscillation around the saddle point, we conclude that there will be corrections of order N^2, i.e. of the same order as the result itself, unlike the case of the O(N) model.

References

Calogero F 1978 *Nuovo Cimento* **B43** 177
Jevicki A 1980 *Brown University preprint HET-429*
Jevicki A and Levine H 1980 *Phys. Rev. Lett.* **44** 1443
Levine H 1980 *Harvard University preprint HUTP-80/A076*
Mehta M L 1967 *Random Matrices* (New York: Academic)
Mondello M and Onofri E 1980 *University of Parma preprint*